THEOREM: THE CONSTANT RULE

Let k be a real number.

$$\cancel{\int k\,dx = x + C} \qquad \int dx = \int 1\,dx = x + C$$

Example 1: Find the indefinite integral.

$$\int -3\,dx = -3\int dx$$

$$= \boxed{-3x + C}$$

THEOREM: THE POWER RULE

Let n be a rational number.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Example 2: Find the following indefinite integrals.

a. $\int x^{-5}\,dx = \dfrac{x^{-5+1}}{-5+1} + C$

$$= -\frac{1}{4}x^{-4} + C$$

$$= \boxed{-\frac{1}{4x^4} + C}$$

b. $\int x^{1/2}dx = \dfrac{x^{3/2}}{3/2} + C$

$= \boxed{\frac{2}{3}x^{3/2} + C}$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is an integrable function and c is a real number, then cf is also integrable and

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Example 3: Find the area of the region bounded by $f(x)=2x^3$, $x=1$, $x=3$, and $y=0$.

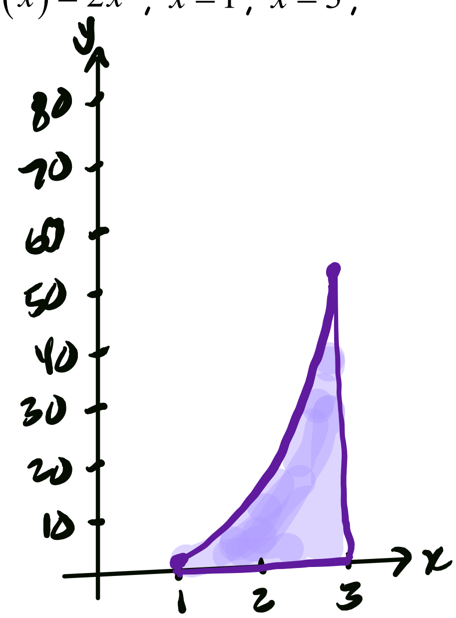
$A = \int_1^3 2x^3\,dx$

$A = \cancel{2}^{1} \cdot \dfrac{x^4}{\cancel{4}_{2}} \Big|_{x=1}^{x=3}$

$A = \frac{1}{2}\left((3)^4 - (1)^4\right)$

$A = \frac{1}{2}(81-1)$

$\boxed{A = 40 \text{ sq. units}}$



THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two integrable functions f and g is itself integrable. Moreover, the antiderivative of $f+g$ (or $f-g$) is the sum (or difference) of the antiderivatives of f and g.

$$\int[f(x)+g(x)]dx=\int f(x)\,dx+\int g(x)\,dx$$

$$\int[f(x)-g(x)]dx=\int f(x)\,dx-\int g(x)\,dx$$

Example 4: Find the indefinite integral.

a. $\int\left(\frac{\sqrt{x}-5x^2}{\sqrt{x}}\right)dx$ $=\int x^{-1/2}(x^{1/2}-5x^2)dx$

$=\int(x^0-5x^{3/2})dx$

$=\int(1-5x^{3/2})dx$

$= x-5\frac{x^{3/2+1}}{3/2+1}+C$

$= x-\cancel{5}^{1}\cdot\frac{2}{\cancel{5}_{1}}x^{5/2}+C$

$=\boxed{x-2x^{5/2}+C}$

b. $\int(x^3+1)^2dx$ $=\int(x^3+1)(x^3+1)dx$

$=\int(x^6+2x^3+1)dx$

$=\boxed{\frac{x^7}{7}+\frac{x^4}{2}+x+C}$

Evil plan

$u=x^3+1$

$du=3x^2\,dx$

Crap... can't get the x^2!

Evil plan 2

Algebra...

$(x^3+1)(x^3+1)$

$(A+B)^2=A^2+2AB+B^2$

c. $\int_3^5 \frac{5+6x+x^2}{5+x}dx$

$= \int_3^5 \left(x+1+\frac{0}{x+5}\right)dx$

Evil Plan

Rewrite $\frac{5+6x+x^2}{5+x}$ by factoring or dividing

$$\begin{array}{r} x+1+\frac{0}{x+5} \\ (x+5)\overline{)x^2+6x+5} \\ -(x^2+5x)\downarrow \\ \hline x+5 \\ -(x+5) \\ \hline 0 \end{array}$$

$(x^2+6x+5)\div(x+5)=x+1$

$= \int_3^5 (x+1)\,dx$

$= \left(\frac{x^2}{2}+x\right|_{x=3}^{x=5}$

$= \left[\frac{(5)^2}{2}+(5)\right]-\left[\frac{(3)^2}{2}+(3)\right]$

$= \frac{25}{2}+2-\frac{9}{2} \rightarrow \; = \boxed{10}$

THEOREM: ANTIDERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \csc x \cot x dx = -\csc x + C$	$\int \sec x \tan x dx = \sec x + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \tan x dx = -\ln\lvert\cos x\rvert + C$	$\int \cot x dx = \ln\lvert\sin x\rvert + C$
$\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$	$\int \csc x dx = -\ln\lvert\csc x + \cot x\rvert + C$

Example 5: Integrate.

a. $\int \sec^2 x dx \; = \boxed{\tan x + C}$

b. $\int(-\csc\theta+\csc\theta\cot\theta)\,d\theta = -(-\ln|\csc\theta+\cot\theta|)+(-\csc\theta)+C$

$= \boxed{\ln|\csc\theta+\cot\theta|-\csc\theta+C}$

c. $\int 3\tan x\,dx = 3(-\ln|\cos x|)+C$

$= \boxed{-3\ln|\cos x|+C}$

$= \boxed{-\ln|\cos^3 x|+C}$ → $= \boxed{\ln\left|\frac{1}{\cos^3 x}\right|+C}$

$= \boxed{\ln|\sec^3 x|+C}$

$(A+B)(A-B)=A^2-B^2$

d. $\int \frac{1}{1+\cos\theta}d\theta \cdot \frac{(1-\cos\theta)}{(1-\cos\theta)} = \int \frac{1-\cos\theta}{1-\cos^2\theta}d\theta$

$u = 1+\cos\theta$

$du = -\sin\theta\, d\theta$

↑ don't have available

$= \int \frac{1-\cos\theta}{\sin^2\theta}$

$= \int\left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin\theta\cdot\sin\theta}\right)d\theta$

$= \int \csc^2\theta\, d\theta - \int \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta$

$= -\cot\theta - \int \csc\theta\cot\theta\, d\theta$

$= -\cot\theta - (-\csc\theta)+C$

$= \boxed{-\cot\theta+\csc\theta+C}$

THEOREM: ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C$$

Example 6: Find the following definite and indefinite integrals.

a. $\int \left(x\sqrt{1-x}\right)dx$

$u = 1-x \rightarrow x = 1-u$

$du = -dx$

$= (-1)\int x(1-x)^{1/2}\,dx\,(-1)$

$= -\int x\,(u)^{1/2}\,du$

$= -\int (1-u)(u)^{1/2}\,du$

$= -\int (u^{1/2} - u^{3/2})\,du$

$= -\left(\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right) + C$

$= -\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$

$= \boxed{-\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C}$

b. $-\frac{1}{4}\int x\left(5-2x^2\right)^5 dx\,(-4)$

$u=5-2x^2$

$du=\underbrace{-4x\,dx}$

$= -\frac{1}{4}\int (u)^5(-4x\,dx)$

$= -\frac{1}{4}\int u^5\,du$

$= -\frac{1}{4}\cdot\frac{u^6}{6}+C$

$= -\frac{1}{24}u^6+C \rightarrow = \boxed{-\frac{1}{24}\left(5-2x^2\right)^6+C}$

c. $\int \cos^2 3x\,dx$

NOTE!

Memorize

$$\begin{cases}\cos^2 A = \dfrac{1+\cos 2A}{2}\\[2ex] \sin^2 A = \dfrac{1-\cos 2A}{2}\end{cases}$$

$= \int \frac{1+\cos\left[2(3x)\right]}{2}\,dx$

$= \int \frac{1}{2}\,dx + \frac{1}{2}\cdot\frac{1}{6}\int \cos 6x\,dx\,(6)$

$u=6x$

$du=6dx$

$= \frac{1}{2}x + \frac{1}{12}\int \cos u\,du$

$= \frac{1}{2}x + \frac{1}{12}\sin u + C$

$= \boxed{\frac{1}{2}x + \frac{1}{12}\sin 6x + C}$

d. $\displaystyle\int_{\pi/4}^{\pi/3} \tan^3 x \sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$u(x) = \tan x$

limits of int. for u

$u(\pi/3) = \tan\frac{\pi}{3} = \sqrt{3}$

$u(\pi/4) = \tan\frac{\pi}{4} = 1$

$= \displaystyle\int_{1}^{\sqrt{3}} (u)^3\,du$

$= \frac{1}{4}u^4 \Big|_{u=1}^{u=\sqrt{3}}$

$= \frac{1}{4}\left[(\sqrt{3})^4 - (1)^4\right]$

$= \frac{1}{4}(9-1)$

$= \boxed{2}$

Theorem: LOG RULE FOR INTEGRATION

Let *u* be a differentiable function of *x*.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Theorem: INTEGRATION RULES FOR EXPONENTIAL FUNCTIONS

Let *u* be a differentiable function of *x*.

1. $\displaystyle\int e^x\,dx = e^x + C$

2. $\displaystyle\int e^u\,du = e^u + C$

3. $\displaystyle\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C,\ a$ is a positive real number, $a \neq 1$

Example 7: Find the following indefinite integrals and evaluate the definite integrals.

$$\begin{array}{r} -5t-9+\frac{17}{2-t} \\ (-t+2)\overline{)5t^2-t-1} \\ -(5t^2-10t) \\ \hline 9t-1 \\ -(9t-18) \\ \hline 17 \end{array}$$

a. $\int \frac{5t^2-t-1}{2-t}\,dt$

$= \int \left[(-5t-9) + \frac{17}{2-t}\right] dt$

$= \int (-5t-9)\,dt + (-1)17\int \frac{1(-1)}{2-t}\,dt$

$u = 2-t$

$du = -dt$

$= -5\frac{t^2}{2} - 9t - 17\int \frac{1}{u}\,du$

$= -\frac{5}{2}t^2 - 9t - 17\ln|u| + C$

$= \boxed{-\frac{5}{2}t^2 - 9t - 17\ln|2-t| + C} \Rightarrow = \boxed{-\frac{5}{2}t^2 - 9t - \ln\left|(2-t)^{17}\right| + C}$

b. $\int \frac{5}{\left(\sqrt{x}\ln x\right)^2}\,dx$

Note: $1000\ln|x-3| = \ln(x-3)^{1000}$

$= 5\int \frac{dx}{x(\ln x)^2}$

$u = \ln x$

$du = \frac{1}{x}dx = \frac{dx}{x}$

$= 5\int \frac{du}{u^2}$

$= 5\int u^{-2}\,du$

$= 5\frac{u^{-1}}{-1} + C$

$= -\frac{5}{u} + C \rightarrow = \boxed{-\frac{5}{\ln x} + C}$

c. $\int \frac{1}{x^{2/3}\left(1+x^{1/3}\right)}dx$ $= 3\int x^{-2/3}\left(1+x^{1/3}\right)^{-1} dx\left(\frac{1}{3}\right)$

$u = 1+x^{1/3}$

$du = \frac{1}{3}x^{-2/3}\,dx$

$= 3\int u^{-1}\,du$

$= 3\ln|u| + C$

$= 3\ln\left|1+x^{1/3}\right| + C$

$= \boxed{\ln\left|\left(1+x^{1/3}\right)^3\right| + C}$

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d. $\int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$ $= \int_1^2 \left(e^x - e^{-x}\right)^{-1}\left(e^x+e^{-x}\right)dx$

$u = e^x - e^{-x}$

$du = \left[e^x - \left(-e^{-x}\right)\right]dx$

$du = \left(e^x + e^{-x}\right)dx$

$u(x) = e^x - e^{-x}$

$u(2) = e^2 - e^{-2}$

$u(1) = e^1 - e^{-1} = e - e^{-1}$

$= \int_{e-e^{-1}}^{e^2-e^{-2}} u^{-1}\,du$

$= \ln|u| \Big|_{u=e-e^{-1}}^{u=e^2-e^{-2}}$

$= \ln\left|e^2 - e^{-2}\right| - \ln\left|e - e^{-1}\right|$

$= \ln\left|\frac{e^2-e^{-2}}{e-e^{-1}}\right|$

$$= \ln \left| \frac{\left(e^2 - \frac{1}{e^2}\right)e^2}{\left(e - \frac{1}{e}\right)e^2} \right|$$

$$= \ln \left| \frac{e^4 - 1}{e^3 - e} \right|$$

$$= \ln \left| \frac{e^4 - 1}{e(e^2-1)} \right|$$

$$= \ln \left| \frac{\cancel{(e^2-1)}(e^2+1)}{e\cancel{(e^2-1)}} \right|$$

$$= \ln \left| \frac{e^2+1}{e} \right|$$

$$= \ln |e^2+1| - \ln e$$

$$= \boxed{\ln(e^2+1) - 1}$$

e. $\int_{\pi/6}^{\pi/4} \sec^2 x dx = \tan x \Big|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{4}}$

$= \tan\frac{\pi}{4} - \tan\frac{\pi}{6}$

$= 1 - \frac{\sqrt{3}}{3}$

$= \boxed{\frac{1}{3}(3-\sqrt{3})}$

f. $-\int_{-\pi/2}^{\pi/2} \sin x \cos^2 x dx \, (-1)$

$u = \cos x$

$du = -\sin x \, dx$

$= -\int_{0}^{0} u^2 du$

$u(x) = \cos x$

$u(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$u(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) = 0$

$= 0$

Same limits of integration!

Theorem: INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

$a \in \mathbb{R}$

Let u be a differentiable function of x, and let $a>0$.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C \qquad \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

Example 7: Find the following indefinite integrals and evaluate the definite integrals.

a. $\displaystyle\int \frac{t}{t^4+2}\,dt = \frac{1}{2}\int \frac{2t\,dt}{(t^2)^2+(\sqrt{2})^2}$

$u = t^2$

$du = 2t\,dt$

$a = \sqrt{2}$

$$= \int \frac{du}{u^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C$$

$$= \boxed{\frac{1}{\sqrt{2}}\arctan\frac{t^2}{\sqrt{2}} + C}$$

complete the square to set up inverse trig result

b. $\int \frac{dx}{\sqrt{5-4x-x^2}}$

$\sqrt{-(x^2+4x+(2)^2)+5+4}$

$=\sqrt{9-(x+2)^2}$

$=\sqrt{(3)^2-(x+2)^2}$

$a=3$, $u=x+2$

$du=dx$

$=\int \frac{du}{\sqrt{(3)^2-u^2}}$

$=\arcsin\frac{u}{3}+C$

$=\boxed{\arcsin\frac{x+2}{3}+C}$

$(b^m)^n=b^{mn}$

$e^{2x}=(e^x)^2$

c. $\int \frac{dx}{\sqrt{e^{2x}-25}}$

$u=e^x$, $a=5$

$\frac{du}{dx}=e^x \rightarrow du=e^x\,dx \rightarrow \frac{du}{e^x}=dx \rightarrow \frac{du}{u}=dx$

$\int \frac{dx}{\sqrt{e^{2x}-25}}=\int \frac{du}{u\sqrt{u^2-5^2}}$

$=\frac{1}{5}\operatorname{arcsec}\frac{|u|}{5}+C$

$=\frac{1}{5}\operatorname{arcsec}\frac{|e^x|}{5}+C$

$=\boxed{\frac{1}{5}\operatorname{arcsec}\frac{e^x}{5}+C}$